Radial Inflow Within Two Flat Disks

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Introduction

THE characteristics of flows confined in narrow gaps formed by two flat circular plates are of interest in industry. Disk-type devices have been widely used in many practical applications.[1,2] Two basic nonswirling types of flow can be produced inside the gap: one is developed by supplying the fluid through a centrally located inlet port, and the other is realized by admitting the flow radially through the circumference. For the former case, the streamlines diverge and the gradual decrease of the local Reynolds number based on the gap height gives rise to the interesting phenomenon of relaminarization.[3] In the latter case, the streamlines converge, resulting in a drastic pressure drop similar to that produced in a converging nozzle.

The present work focuses on the second type of flow. Previous theoretical studies have concentrated on solutions to the integral equations of motion.[1,4-6] Lee and Lin[7] attempted a differential solution of the linearized momentum equation to obtain the radial pressure distribution. Their method of treating the equation led them to an expression for pressure that was amenable only to a numerical treatment. Nonetheless, the results obtained have shown a good agreement with the experiment of Ref. 6. In a recent paper,[8] the author, convinced also that only a numerical result for the linearized momentum equation was possible, simplified further the inertial terms. With the additional simplification, he was able to obtain a closed-form solution for the pressure which agreed closely with the numerical results of Ref. 7.

In the present Note, closed-form solutions of the linearized momentum equations of Lee and Lin[7] are presented. The inviscid and creeping-flow solutions are shown to be the asymptotes of the general problem. The results have been found to confirm the observation.

Analysis and Discussion

Consider the nonswirling, steady laminar flow between the disks shown in Fig. 1. Under the assumption that the flow is purely radial, the equations of motion, in nondimensional form,[8] are

Continuity:

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left[\bar{V}_r \cdot \bar{r}\right] = 0 \tag{1}$$

r momentum:

$$\bar{V}_r\frac{\partial \bar{V}_r}{\partial \bar{r}} = -\frac{\mathrm{d}(\overline{\Delta P})}{\mathrm{d}\bar{r}} + \frac{1}{\bar{R}e}\frac{\partial^2 \bar{V}_r}{\partial \bar{z}^2} \tag{2}$$

where $\bar{r} = r/R_o$, $\bar{z} = z/h$, $\bar{V}_r = V_r/V_o$, $Re = V_o h/\nu$, $\bar{R}e = \xi Re$, $\overline{\Delta P} = (P_o - P)/\rho V_o^2$, and $\xi = h/R_o$.

From continuity

$$\bar{V}_r \cdot \bar{r} = \Phi(\bar{z}) \tag{3}$$

where Φ, as Eq. (3) shows, is an arbitrary function of $\bar{z}$. Inserting Eq. (3) into Eq. (2) yields

$$-\frac{\Phi^2}{\bar{r}^3} = -\frac{\mathrm{d}(\overline{\Delta P})}{\mathrm{d}\bar{r}} + \frac{1}{\bar{R}e\bar{r}}\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2} \tag{4}$$

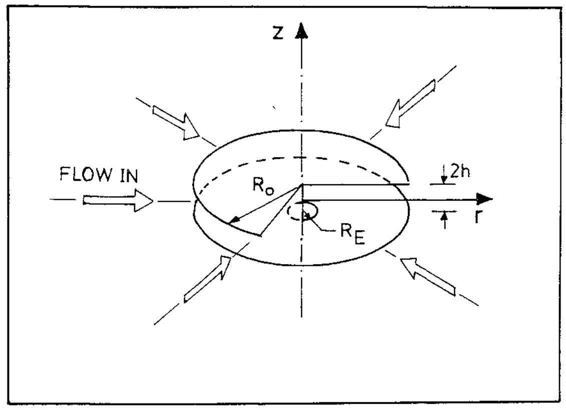

Fig. 1 Geometry of problem.

To obtain a closed-form solution, Eq. (2) will be linearized assuming[7]

$$\bar{V}_r\frac{\partial \bar{V}_r}{\partial \bar{r}} \cong -\frac{1}{\bar{r}}\frac{\partial \bar{V}_r}{\partial \bar{r}}$$

Then, Eq. (4) becomes

$$\frac{\Phi}{\bar{r}^3} = -\frac{\mathrm{d}(\overline{\Delta P})}{\mathrm{d}\bar{r}} + \frac{1}{\bar{R}e\bar{r}}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\bar{z}^2} \tag{5}$$

Integration of Eq. (5) with $\bar{r}$ (from $\bar{r}_1$ to 1.0) yields

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\bar{z}^2} - \frac{\Phi}{2}\left(\frac{\bar{R}e}{\ell n\bar{r}_1}\right)\frac{(\bar{r}_1^2 - 1)}{\bar{r}_1^2} = \frac{\bar{R}e}{\ell n\bar{r}_1}\overline{\Delta P}(\bar{r}_1) \tag{6}$$

The particular solution to Eq. (6) is

$$\Phi_p(\bar{z}) = -\frac{2\bar{r}_1^2\overline{\Delta P}}{(\bar{r}_1^2 - 1)}$$

and the homogeneous solution is

$$\Phi_h(\bar{z}) = Ae^{\lambda\bar{z}} + Be^{-\lambda\bar{z}}$$

where

$$\lambda = +\sqrt{\frac{\bar{R}e(\bar{r}_1^2 - 1)}{2\bar{r}_1^2\,\ell n\bar{r}_1}}$$

Then,

$$\Phi(\bar{z}) = Ae^{\lambda\bar{z}} + Be^{-\lambda\bar{z}} - \frac{2\bar{r}_1^2\overline{\Delta P}}{(\bar{r}_1^2 - 1)} \tag{7}$$

where A and B are general constants. Using the no-slip velocity condition at $\bar{z} = 1.0$ and the symmetrical velocity requirement at $\bar{z} = 0$, i.e.,

$$\Phi(\bar{z} = 1.0) = 0$$

$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}\bar{z}}\right|_{\bar{z}=0} = 0$$

respectively, the two constants in Eq. (7) are evaluated to give

$$\Phi(\bar{z}) = \frac{2\bar{r}_1^2\overline{\Delta P}}{\bar{r}_1^2 - 1}\left[\frac{\cosh(\lambda\bar{z})}{\cosh(\lambda)} - 1\right] \tag{8}$$

Application of continuity between $\bar{r}$ and the inlet ($\bar{r} = 1.0$) gives

$$1 = \int_0^1 \bar{V}_r(\bar{r} = \bar{r}_1)\,\mathrm{d}\bar{z}$$

or

$$1 = \frac{2\bar{r}_1^2\overline{\Delta P}}{\bar{r}_1^2 - 1}\int_0^1\left[1 - \frac{\cosh(\lambda\bar{z})}{\cosh(\lambda)}\right]\mathrm{d}\bar{z} \tag{9}$$

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Evaluation of the integral in Eq. (9) yields

$$\overline{\Delta P} = \frac{1}{2}\left(\frac{\bar{r}_1^2 - 1}{\bar{r}_1^2}\right)\left[\frac{\lambda}{\lambda - \tanh(\lambda)}\right] \tag{10}$$

From Eqs. (3) and (8), the normalized velocity is given by

$$\frac{\bar{V}_r}{\bar{V}_{r\,\mathrm{mean}}} = \left[\frac{\lambda}{\lambda - \tanh(\lambda)}\right]\left[1 - \frac{\cosh(\lambda\bar{z})}{\cosh(\lambda)}\right] \tag{11}$$

where $\bar{V}_{r\mathrm{mean}}$ is the local mean velocity.

Applying l'Hôspital's rule to the velocity equation, one gets

$$\lim_{\bar{R}e \to \delta}\left[\frac{\bar{V}_r}{\bar{V}_{r\,\mathrm{mean}}}\right] = \lim_{\bar{R}e \to \delta}\left\{\frac{\dfrac{d^2}{d\bar{R}e^2}\Big(\cosh(\lambda) - \lambda[\cosh(\lambda\bar{z})]\Big)}{\dfrac{d^2}{d\bar{R}e^2}\Big([\lambda - \tanh(\lambda)]\cosh(\lambda)\Big)}\right\}$$

$$\lim_{\bar{R}e \to \delta}\left[\frac{\bar{V}_r}{\bar{V}_{r\mathrm{mean}}}\right]$$

$$= \lim_{\bar{R}e \to \delta}\left\{\frac{2\left[\dfrac{\tanh(\lambda)}{\lambda} - \bar{z}\,\dfrac{\sinh(\lambda\bar{z})}{\lambda\cosh(\lambda)}\right] - \bar{z}^2\,\dfrac{\cosh(\lambda\bar{z})}{\cosh(\lambda)} + 1}{1 + \dfrac{\tanh(\lambda)}{\lambda}}\right\}$$

Using the following relationships:

$$\lim_{\bar{R}e \to \delta}\left\{\frac{\tanh(\lambda)}{\lambda}\right\} = \begin{array}{l} 1 \text{ if } \delta = 0 \\ 0 \text{ if } \delta = \infty \end{array}$$

$$\lim_{\bar{R}e \to \delta}\left\{\frac{\sinh(\lambda\bar{z})}{\lambda\cosh(\lambda)}\right\} = \begin{array}{l} z \text{ if } \delta = 0 \\ 0 \text{ if } \delta = \infty \end{array}$$

$$\lim_{\bar{R}e \to \delta}\left\{\frac{\cosh(\lambda\bar{z})}{\cosh(\lambda)}\right\} = \begin{array}{l} 1 \text{ if } \delta = 0 \\ 0 \text{ if } \delta = \infty \end{array}$$

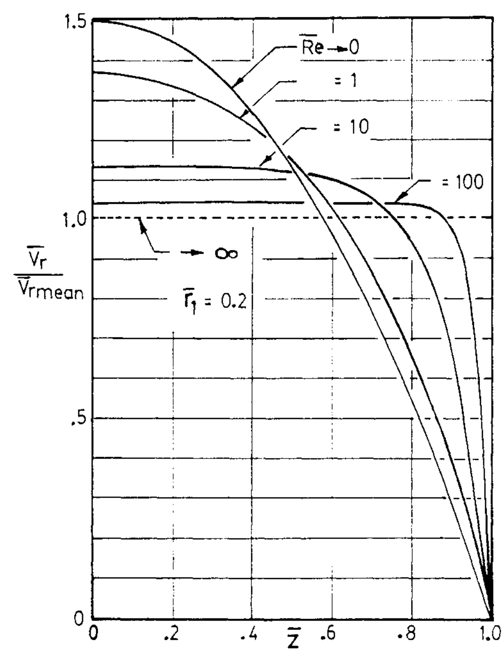

Fig. 2 Radial velocity distributions for different Reynolds numbers.

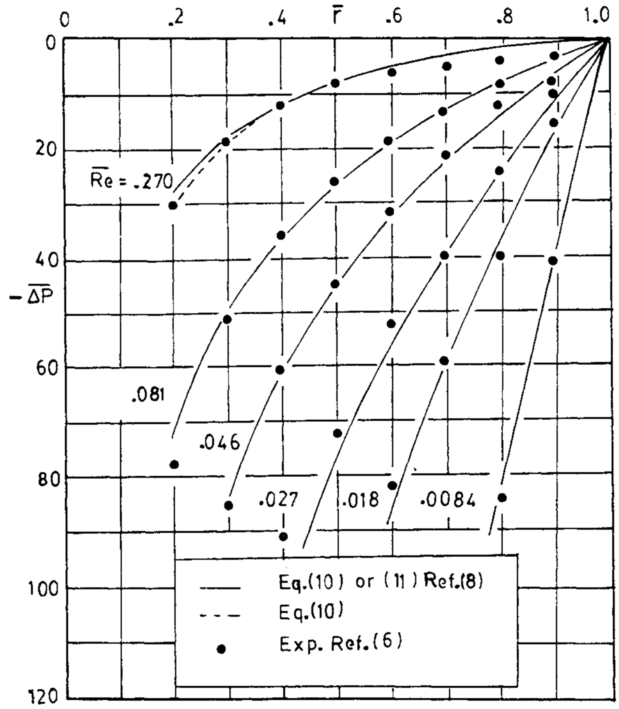

Fig. 3 Static pressure distributions.

the limiting cases for the velocity are

$$\lim_{\bar{R}e \to \delta}\left[\frac{\bar{V}_r}{\bar{V}_{r\mathrm{mean}}}\right] = \begin{cases} 3/2(1 - \bar{z}^2) & \text{for } \delta = 0 \\ 1 & \text{for } \delta = \infty \end{cases}$$

Similarily, the pressure equation (11) gives[7]

$$\lim_{\bar{R}e \to \delta}\left[\overline{\Delta P}\right] = \begin{cases} \infty & \text{for } \delta = 0 \\ \dfrac{1}{2}\left(\dfrac{\bar{r}_1^2 - 1}{\bar{r}_1^2}\right) & \text{for } \delta = \infty \end{cases} \tag{12}$$

Therefore, creeping and inviscid flows are the asymptotes of the present, more general problem. From Fig. 2, it is evident that for high Reynolds numbers, the velocity remains practically flat over a large portion of the gap, dropping abruptly to zero near the wall, thus showing boundary-layer characteristics. For low Reynolds numbers, viscous forces dominate the inertia forces, shear waves from the wall penetrate further into the flowfield in the $\bar{z}$ direction, and the flow exhibits the familiar Poisseuille parabolic profile. Hence, the velocity distribution given in Ref. 8 applies only in the low Reynolds number regime. Inspite of the simplifications introduced in Ref. 8, the pressure drop equation retains the main effects of the asymptotic flow. The difference between Eq. (11) and that presented in Ref. 8 is not significant for large and low Reynolds numbers. For intermediate values of $\bar{R}e$, the difference is larger but never exceeds 12.50%. Figure 3 shows close agreement of the present pressure-drop equation with that given by the author in Ref. 8, and the experiments of Ref. 6 (the results were taken from Ref. 7). A good agreement was expected since the experiments of Ref. 6 are in the low Reynolds number range.

Acknowledgment

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Flow Refraction by an Uncoupled Shock and Reaction Front

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Nomenclature

C_p = specific heat
E = activation energy
M = Mach number
p = pressure
Q = heat release
$\hat{Q}$ = $(1 - \beta_2)Q$
$\bar{Q}$ = Q/RT_f
R = gas constant
T = temperature
u,v = velocity components
β = reaction progress variable
γ = specific heat ratio
δ = shock angle
θ = wedge angle
$(\ )_f$ = fresh gas
$(\ )_n$ = normal to front
$(\ )_1$ = upstream of front
$(\ )_2$ = downstream of front

Introduction

CONSIDER an oblique detonation wave supported, for example, by a wedge. Given the freestream conditions, the heat released in the wave, and the wedge (flow-deflection) angle, we can ask if a wave angle can be found so that the Rankine-Hugoniot conditions are satisfied. Gross[1] has shown that when the freestream conditions and the heat released are prescribed, this is possible provided the wedge angle is smaller than some critical value. Moreover, additional conditions, arising from consideration of the wave structure, define a *minimum* wedge angle. The permissible range of wedge angles, determined in this way, is of practical concern in any flow configuration in which oblique detonations are expected, e.g., in Ref. 2.

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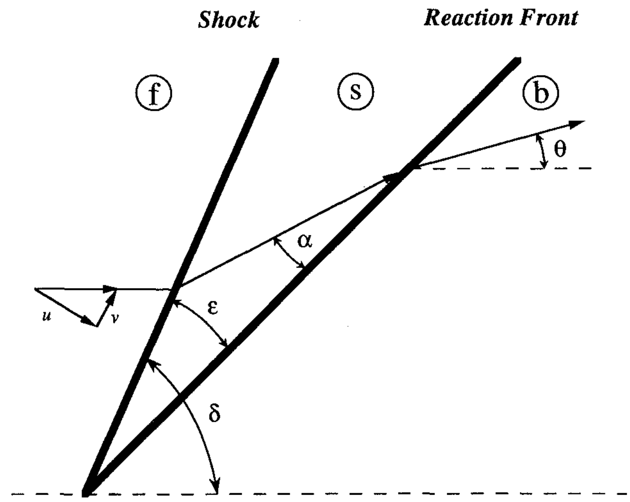

Fig. 1 Refraction by dual fronts.

Recently, Fujiwara et al.[3] have carried out numerical simulations of flow past wedges and have computed steady solutions (as the late time limit of unsteady ones) characterized by oblique detonations. The calculations employ a 2-step kinetic scheme in which the first (irreversible) step has an Arrhenius temperature dependence with a large activation energy. The associated one-dimensional steady detonation structure is characterized by a nearly isothermal induction zone behind the lead shock, terminated by a thin reaction zone or fire. In a genuine oblique detonation wave, remote from any disturbance (such as a blunted wedge nose), this fire will be parallel to the shock, and the structure will be one-dimensional. However, in some cases, Fujiwara et al. found steady solutions in which the shock and the fire are uncoupled in the sense that they drift apart as the distance from the symmetry axis increases. The refraction of the flow is then generated by a two-dimensional structure, and qualitative conclusions drawn from calculations similar to those of Gross must be modified. In the present paper, we approximate the structure by two fronts, the shock and the fire, inclined relative to each other, and we calculate the refraction that they generate. We find that such dual fronts are more effective than an oblique detonation wave in turning the flow (i.e., larger wedge angles are possible).

Flow Refraction by a Single Front

Consider a uniform flow which passes through a single front ($\epsilon = 0$ in Fig. 1). Then the Rankine-Hugoniot conditions are

$$\rho_1 u_1 = \rho_2 u_2$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{1}$$

$$v_1 = v_2$$

$$C_p T_1 + \frac{1}{2}(u_1^2 + v_1^2) + Q = C_p T_2 + \frac{1}{2}(u_2^2 + v_2^2)$$

The equation of state is

$$p = \rho R T \tag{2}$$

valid on each side of the front, and geometrical conditions are

$$\tan\delta = u_1/v_1, \qquad \tan(\delta - \theta) = u_2/v_2 \tag{3}$$